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The character of equilibrium of a heavy viscous incompressible, finitely conducting, rotating fluid in the presence of a vertical magnetic field

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The character of equilibrium of heavy, viscous, incompressible, finitely conducting, rotating fluid in the presence of magnetic field along the direction of gravitational field has been investigated. It has been shown that the solution is characterized by a variational principle. Based on the existence of variational principle an approximate solution has been derived for the case of a fluid having exponentially varying density in the vertical direction. Due to finite conductivity of the fluid it is found that potentially stable or unstable configuration retains its character. The growth rate of disturbance has been obtained corresponding to long and short wave lengths. It is shown that for short wave lengths the growth rate of disturbance is independent of the magnitude of rotation and finite electrical conductivity. Finally the special case in which the waves arise in the absence of buoyancy forces has been treated.

1. Introduction

Lord Rayleigh (1883) was one of the first to investigate the equilibrium of a stratified inviscid fluid and found that the equilibrium of a horizontal layer of a heavy incompressible fluid of a variable density ρ_0 is stable or unstable according as $d\rho_0/dz$ is everywhere negative or is anywhere positive. Chandrasekhar (1955) studied the character of the equilibrium of an incompressible, heavy viscous fluid of variable density and observed that in the stable case, the fluid oscillates about the mean position with an amplitude which decays exponentially at a rate which increases with increasing viscosity. Hide (1955) studied further the effect of magnetic field and found that magnetic field, considerably stabilizes the system. It is possible to have oscillatory motion in the presence of magnetic field even if the system is thoroughly unstable.

Hide (1956) also considered the effect of uniform rotation on the character of the equilibrium of a stratified fluid of finite depth of exponentially varying density and found that the combined influence of viscosity and rotation instabilizes the system.

Ariel (1971) investigated the effect of a uniform magnetic field along the direction of g on the character of equilibrium of a heavy, viscous, incompressible, infinitely conducting, rotating fluid of variable density and it is shown that the solution is characterized by a variational principle. Based on variational principle, he has shown that both magnetic field and the coriolis forces tend to stabilize the system separately.

In a more realistic physical situation one must take into account the finite conductivity of the medium. We thus, investigate the effect of the finite conductivity of the medium on the equilibrium of a heavy viscous, incompressible, rotating fluid of variable density in the presence of a vertical magnetic field.

2. Basic Equations

The equation of motion to the problem under consideration is

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho(\boldsymbol{u}.\nabla)\boldsymbol{u} = \nabla\cdot[-p+\mu(\nabla\boldsymbol{u}+\boldsymbol{u}\nabla)]+\kappa/4\pi[(\nabla\times\boldsymbol{H})\times\boldsymbol{H}]+2\rho(\boldsymbol{u}\times\boldsymbol{\Omega})+\rho\boldsymbol{g}, \tag{1}$$

where

ρ density of fluid at point P at time t

p pressure at any point

$\boldsymbol{u}$ velocity at any point (u, v, w)

$\boldsymbol{g}$ acceleration due to gravity with component $-g$ in the z-direction

κ coefficient of magnetic permeability

$\boldsymbol{J}$ electrical current density

$\boldsymbol{H}$ uniform magnetic field pervading the fluid configuration, $(0, 0, H)$

μ coefficient of viscosity.

The equation of conservation of matter of an incompressible fluid is

$$\nabla\cdot\boldsymbol{u} = 0. \quad \ldots \tag{2}$$

Also, since the fluid is incompressible, the density of any fluid particle should remain the same throughout the motion, therefore

$$\frac{\partial\rho}{\partial t}+(\boldsymbol{u}\cdot\nabla)\rho = 0 \tag{3}$$

Maxwell equations are

$$\nabla\times \boldsymbol{H} = 4\pi \boldsymbol{J} \quad \ldots \quad (4)$$

$$\nabla\times \boldsymbol{E} = -\kappa \frac{\partial \boldsymbol{H}}{\partial t} \quad \ldots \quad (5)$$

$$\nabla\times \boldsymbol{H} = 0. \quad \ldots \quad (6)$$

The relation between the coefficient of electrical conductivity (σ), current density, electrical field and magnetic field is given by

$$\boldsymbol{J} = \sigma(\boldsymbol{E}+\kappa \boldsymbol{u}\times \boldsymbol{H}) \quad \ldots \quad (7)$$

Now eliminating $\boldsymbol{J}$ and $\boldsymbol{E}$ between eqs. (4), (5) and (7) by operating on eqs. (4) and (7) with curl, we have

$$\frac{\partial \boldsymbol{H}}{\partial t} - \nabla(\boldsymbol{u}\times \boldsymbol{H}) = \eta\nabla^2 \boldsymbol{H}, \quad \ldots \quad (8)$$

where

$$\eta = \frac{1}{4\pi\kappa\sigma}. \quad \ldots \quad (9)$$

Let us consider the effect of a small perturbation upon the static equilibrium configuration which produces a velocity field $\boldsymbol{u}$ (components u, v, w). Let the corresponding perturbations in the density, pressure, coefficient of viscosity and magnetic field be $\delta\rho$, δp, $\delta\mu$, and $\boldsymbol{h}$ respectively i.e.,

$$\rho = \rho_0(z)+\delta\rho(x, y, z, t) \quad \ldots \quad (10)$$

$$p = p_0(z)+\delta p(x, y, z, t) \quad \ldots \quad (11)$$

$$\mu = \mu_0(z)+\delta\mu(x, y, z, t) \quad \ldots \quad (12)$$

$$\boldsymbol{H} = H_0+\boldsymbol{h}(x, y, z, t). \quad \ldots \quad (13)$$

The linear equations governing these perturbations are

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial \delta p}{\partial x} + \kappa \frac{H_0}{4\pi}\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right) + 2\rho_0 v\Omega + \mu_0\nabla^2 u + \frac{\partial \mu_0}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \quad \ldots \quad (14)$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial \delta p}{\partial y} + \kappa \frac{H_0}{4\pi}\left(\frac{\partial h_y}{\partial z} - \frac{\partial h_z}{\partial y}\right) - 2\rho_0 u\Omega + \mu_0\nabla^2 v + \frac{\partial \mu_0}{\partial z}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \quad \ldots \quad (15)$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial \delta p}{\partial z} + \mu_0\nabla^2 w + 2\frac{\partial \mu_0}{\partial z}\frac{\partial w}{\partial z} - g\delta p. \quad \ldots \quad (16)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \ldots \quad (17)$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0 \quad \ldots \quad (18)$$

$$\frac{\partial}{\partial t}\delta\rho+w\frac{d\rho}{dz}=0 \quad \ldots \quad (19)$$

$$\frac{\partial h}{\partial t}-H_0\frac{\partial u}{\partial z}=\eta\nabla^2 h. \quad \ldots \quad (20)$$

Analysing the distrubance into normal modes, we seek solutions, whose dependence on x, y, and t is given by exp. (ik_x+ik_y+nt), where k_x and k_y are the horizontal components of the wave-vector k and n is the rate at which the system departs from equilibrium. For solutions having this dependence on x y and t, eqs. (14) to (20) becomes

$$ik_x\delta p=-n\rho_0 u+(\kappa H_0/4\pi)(Dh_x-ik_xh_z)+2\rho_0 v\Omega+\mu_0(D^2-k^2)u$$
$$+(ik_xw+Du)D\mu_0 \quad \ldots \quad (21)$$

$$ik_y\delta p=-n\rho_0 v+\kappa H_0/4\pi(Dh_y-ik_yh_z)-2\rho_0 u\Omega+\mu_0(D^2-k^2)v$$
$$+(ik_yw+Dv)D\mu_0 \quad \ldots \quad (22)$$

$$D\delta p=-n\rho_0 w+\mu_0(D^2-k^2)w+2D\mu_0 Dw-g\delta\rho \quad \ldots \quad (23)$$

$$ik_xu+ik_yv=-Dw \quad \ldots \quad (24)$$

$$ik_xh_x+ik_yh_y=-Dh_z \quad \ldots \quad (25)$$

$$n\delta\rho=-wD\rho_0 \quad \ldots \quad (26)$$

$$[n-\eta(D^2-k^2)]\boldsymbol{h}=H_0D\boldsymbol{u}\,. \quad \ldots \quad (27)$$

where D denotes the differentiation with respect to z and $k=(k_x^2+k_y^2)^{\frac{1}{2}}$ is total wave number of disturbance.

Multiplying eqs. (21) and (22) by $-ik_x$ and $-ik_y$ respectively and then adding and making use of eqs. (24) and (25), we find that

$$k^2\delta p=-n\rho_0 Dw+(\kappa H_0/4\pi)(D^2-k^2)h_z-2\rho_0\Omega\zeta+\mu_0(D^2-k^2)Dw+D\mu_0(D^2+k^2)w \quad \ldots \quad (28)$$

where

$$\zeta=ik_xv-ik_yu \quad \ldots \quad (29)$$

is the z-component of the vorticity vector.

Inserting the value of $\delta\rho$ from eq. (26) in eq. (23), we obtain

$$D\delta p=-n\rho_0 w+\mu_0(D^2-k^2)w+2(D\mu_0 Dw)+\frac{g}{n}(D\rho_0)w \quad \ldots \quad (30)$$

Now eliminating δp between eqs. (28) and (30), we obtain

$$n[k^2\rho_0 w - D(\rho_0 Dw)] + (\kappa H_0/4\pi)(D^2-k^2)Dh_z - (gk^2/n)(D\rho_0)w - D(2\rho_0\Omega\zeta) + \mu_0(D^2-k^2)^2 w + 2D\mu_0(D^2-k^2)Dw + D^2\mu_0(D^2+k^2)w = 0. \quad \ldots \quad (31)$$

Multiplying (21) and (22) by $-ik_x$ and ik_y respectively, and adding, on making use of (29), we have

$$[n\rho_0 - \mu_0(D^2-k^2) - D\mu_0 D]\zeta - (\kappa H_0/4\pi)D\xi = 2\rho_0\Omega Dw \quad \ldots \quad (32)$$

where

$$\xi = ik_x h_y - ik_y h_x. \quad \ldots \quad (33)$$

The z-components of eq. (27) is

$$[n - \eta(D^2-k^2)]h_z = H_0 Dw. \quad \ldots \quad (34)$$

Multiplying x and y components of eq. (27) by $-ik_y$ and ik_x and adding, we have

$$[n - \eta(D^2-k^2)]\xi = H_0 D\zeta. \quad \ldots \quad (35)$$

3. Boundary Conditions

At a free surface the kinematical condition is that a particle which is at the surface at one instant of time will remain there indefinitely. However, following Rayleigh, we assume that there are no surface waves (potential energy of the disturbation free case is not a major consideration in this case) and take

$$w = 0 \quad \text{on a free surface} \quad \ldots \quad (36)$$

The existence of tangential stresses within the fluid leads to another condition, namely, that p_{xz} and p_{yz} must vanish on a free surface.

Now the tangential stresses can be shown to be

$$p_{xz} = \mu_0\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] + \kappa\frac{H_0}{4\pi}h_x = \mu_0[ik_x w + Du] + \kappa\frac{H_0}{4\pi}h_x \quad \ldots \quad (37)$$

$$p_{yz} = \mu_0\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] + \kappa\frac{H_0}{4\pi}h_y = \mu_0[ik_y w + Dv] + \kappa\frac{H_0}{4\pi}h_y. \quad \ldots \quad (38)$$

Thus $ik_x p_{xz} + ik_y p_{yz}$ and $-ik_y p_{xz} + ik_x p_{yz}$ must vanish, so that

$$\mu_0(D^2+k^2)w + k\frac{H_0}{4\pi}Dh_z = 0 \quad \ldots \quad (39)$$

and

$$\mu_0 D\zeta + k\frac{H_0}{4\pi}\xi = 0. \quad \ldots \quad (40)$$

For the sake of mathematical tractability, we shall be following Hide (1955) and assume that $D^2w = 0$ at a free surface. Hence, by (36) and (39) the remaining conditions, become

$$D^2w = 0 = Dh_z \quad \text{on the free surface} \quad \ldots (41)$$

From eq. (40), it follows that

$$D\zeta = \xi = 0 \quad \text{on the free surface} \quad \ldots (42)$$

In case the fluid terminates at a rigid boundary, $\boldsymbol{u}$ can have no normal component; because the fluid cannot slip relative to the boundary due to viscosity. Thus

$$\boldsymbol{u} = 0 \quad \text{on a rigid boundary.} \quad \ldots (43)$$

Since u and v both vanish, ik_xu+ik_yv, by eq. (24) is equal to Dw, hence

$$w = 0 = Dw = 0 \quad \text{on a rigid boundary.} \quad \ldots (44)$$

The boundary condition on the normal component of vorticity vector, can be deduced by making use of the equation

$$\zeta = ik_xv-ik_yu \quad (45)$$

and from eq. (43), it follows that

$$\zeta = 0 \quad \text{on a rigid surface.} \quad (46)$$

Further conditions arise when we consider the behaviour of the magnetic and electric fields of the boundary. If the fluid is bounded by an ideal conductor, no disturbance within the fluid can charge E and H outside the fluid. Since surface charges and surface currents can only allow discontinuities in E_z, h_x and h_y, we must require that

$$E_x = E_y = h_z = 0, \quad \ldots (47)$$

$$E_x = k[\eta ik_yh_z-\eta Dh_y-vH_0], \quad \ldots (48)$$

$$E_y = k[\eta Dh_x-\eta ik_xh_z+uH_0]. \quad \ldots (49)$$

Thus $ik_xE_x+ik_yE_y$ and $ik_xE_y-ik_yE_x$ must vanish, so that

$$(\eta D\xi+H_0\zeta) = 0 \quad \ldots (50)$$

and

$$[\eta(D^2-k^2)h_z+H_0Dw] = 0 \quad \ldots (51)$$

Now on a rigid boundary $Dw = 0$ and $\xi = 0$.

Therefore,

$$D\xi = 0 \quad \text{and} \quad h_z = 0 \quad \ldots (52)$$

at a surface bounded by an ideal conductor.

4. A Variational Principle

Let n_i and n_j denote two of the characteristic values of n and let the solutions belonging to these characteristic values be distinguished by the subscripts i and j.

Now consider the eq. (31) for the characteristic value n_i and after multiplication by w_j (belonging to n_j) integrate over the vertical extent of fluid ($0 \leqslant z \leqslant d$), we have

$$n_i \int_0^d [k^2\rho_0 w_i - D(\rho_0 Dw_i)]w_j dz + (\kappa H_0/4\pi)\int_0^d w_j(D^2-k^2)Dh_i dz - \frac{gk^2}{n_i}\int_0^d D\rho_0 w_i w_j dz$$

$$-\int_0^d D(2\rho_0\Omega\zeta_i)dz + \int_0^d \{\mu_0(D^2-k^2)^2 w_i + 2D\mu_0(D^2-k^2)Dw_i$$

$$+D^2\mu_0(D^2-k^2)w_i\}w_j dz = 0. \quad \dots \quad (53)$$

The magnetic term in eq. (53)

$$I^1 = \kappa\frac{H_0}{4\pi}\int_0^d w_j(D^2-k^2)Dh_i dz = -\kappa\frac{H_0}{4\pi}\int_0^d Dw_j(D^2-k^2)h_i dz. \quad \dots \quad (54)$$

Substituting for Dw_j from eq. (34) in eq. (54), we have

$$I^1 = \frac{\kappa}{4\pi}(n_j+\eta k^2)k^2\int_0^d h_i h_j dz + \frac{\kappa}{4\pi}(n_j+2\eta k^2)\int_0^d Dh_i Dh_j dz + \kappa\frac{(\eta k^2)}{4\pi k^2}\int_0^d D^2h_i D^2h_j dz$$

$$= 0 \quad \dots \quad (55)$$

Similarly, the partial integration of Ω terms yields

$$I^{11} = -\int_0^d D(2\rho_0\Omega\zeta_i)w_j dz = \int_0^d 2\rho_0\Omega\zeta_i Dw_j dz. \quad \dots \quad (56)$$

Substituting for Dw_j from eq. (32) in eq. (56), we have

$$I^{11} = ng\int_0^d \rho_0\zeta_i\zeta_j dz + \int_0^d \mu_0(k^2\zeta_i\zeta_j + D\zeta_i D\zeta_j)dz + \kappa\frac{H_0}{4\pi}\int_0^d \xi_j D\zeta_i dz. \quad \dots \quad (57)$$

Inserting the value of $D\zeta_i$ from eq. (35) in eq. (57), we get,

$$I^{11} = ng\int_0^d \rho_0\zeta_i\zeta_j dz + \int_0^d \mu_0(k^2\zeta_i\zeta_j + D\zeta_i D\zeta_j)dz + \kappa\frac{(\eta_i+\eta k^2)}{4\pi}\int_0^d \xi_i\xi_j dz + \kappa\frac{\eta k^2}{4\pi k^2}\int_0^d D\xi_i D\xi_j dz$$

$$\dots \quad (58)$$

Substituting the values of I^1 and I^{11} in the eq. (53), we obtain

$$n_i(I_1+I_9) - (gk^2/n_i)I_2 + I_3 + I_8 + n_j(I_4+I_5+I_7) + \eta k^2(I_4+2I_5+I_6+I_9+I_{10}) = 0$$

$$\dots \quad (59)$$

where

$$I_1 = \int_0^d \rho_0(k^2 w_i w_j + Dw_i Dw_j)dz \quad \ldots \quad (60)$$

$$I_2 = \int_0^d D\rho_0 w_i w_j dz \quad \ldots \quad (61)$$

$$I_3 = \int_0^d \{\mu_0(D^2-k^2)w_i + 2D\mu_0(D^2-k^2)Dw_i + D^2\mu_0(D^2+k^2)w_j\}w_j dz \quad \ldots \quad (62)$$

$$I_4 = \kappa \frac{k^2}{4\pi} \int_0^d h_i h_j dz \quad \ldots \quad (63)$$

$$I_5 = \frac{\kappa}{4\pi} \int_0^d Dh_i Dh_j dz \quad \ldots \quad (64)$$

$$I_6 = \frac{\kappa}{4\pi k^2} \int_0^d D^2 h_i D^2 h_j dz \quad \ldots \quad (65)$$

$$I_7 = \int_0^d \rho_0 \zeta_i \zeta_j dz \quad \ldots \quad (66)$$

$$I_8 = \int_0^d \mu_0(k^2 \zeta_i \zeta_j + D\zeta_i D\zeta_j)dz \quad \ldots \quad (67)$$

$$I_9 = \frac{\kappa}{4\pi} \int_0^d \xi_i \xi_j dz \quad \ldots \quad (68)$$

$$I_{10} = \frac{\kappa}{4\pi k^2} \int_0^d D\xi_i D\xi_j dz. \quad \ldots \quad (69)$$

If we write $i = j$ in eq. (59), we obtain

$$n(I_1+I_4+I_5+I_7+I_9)-(gk^2/n)I_2+\eta k^2(I_4+2I_5+I_6+I_9+I_{10})+I_3+I_8 = 0. \quad \ldots \quad (70)$$

Consider the change δn in n consequent upon a first order arbitrary variation δw, δh, $\delta\zeta$ and $\delta\xi$ in w, h, ζ and ξ respectively that satisfy the boundary conditions of the problem, we have to the first order of approximation

$$\begin{aligned}-\delta n[I_1+I_4+I_5+I_7+I_9+(gk^2/n^2)I_2] &= n(\delta I_1+\delta I_4+\delta I_5+\delta I_7+\delta I_9)\\ &-(gk^2/n)\delta I_2+\delta I_3+\delta I_8+\eta k^2(\delta I_4+2\delta I_5+\delta I_6+\delta I_9+\delta I_{10})\end{aligned} \quad \ldots \quad (71)$$

where δI is the corresponding change in I.

After one or more integrations by parts we find that these variations are given by

$$\tfrac{1}{2}\delta I_1 = \int_0^d [\rho_0 k^2 w - D(\rho_0 Dw)]\delta w dz \quad \ldots \quad (72)$$

$$\tfrac{1}{2}\delta I_2 = \int_0^d D\rho_0 w dz \quad \dots \quad (73)$$

$$\tfrac{1}{2}\delta I_3 = \int_0^d \{\mu_0(D^2-k^2)^2 w + 2D\mu_0(D^2-k^2)Dw + D^2\mu_0(D^2+k^2)w\}\delta w dz \quad \dots \quad (74)$$

$$\tfrac{1}{2}\delta I_4 = \kappa \frac{k^2}{4\pi} \int_0^d h\, \delta h dz \quad \dots \quad (75)$$

$$\tfrac{1}{2}\delta I_5 = -\frac{\kappa}{4\pi} \int_0^d D^2 h\, \delta h\, dz \quad \dots \quad (76)$$

$$\tfrac{1}{2}\delta I_6 = -\frac{\kappa}{4\pi k^2} \int_0^d D^4 h\, \delta h\, dz \quad \dots \quad (77)$$

$$\tfrac{1}{2}\delta I_7 = \int_0^d \rho_0 \zeta\, \delta\zeta\, dz \quad \dots \quad (78)$$

$$\tfrac{1}{2}\delta I_8 = \int_0^d \{\mu_0[k^2\zeta - D(\mu_0 D\zeta)]\}\delta\zeta\, dz \quad \dots \quad (79)$$

$$\tfrac{1}{2}\delta I_9 = \frac{\kappa}{4\pi} \int_0^d \xi \delta\xi\, dz \quad \dots \quad (80)$$

$$\tfrac{1}{2}\delta I_{10} = -\frac{\kappa}{4\pi k^2} \int_0^d D^2\xi\, \delta\xi\, dz. \quad \dots \quad (81)$$

Differentiating partially eqs. (32), (34) and (36), we get the following equations

$$[(n\rho_0+\mu_0 k^2)-\mu_0 D^2 - D\mu_0 D]\delta\zeta + \delta n \rho_0 \zeta - (H_0\, \kappa/4\pi)D\delta\xi = 2\rho_0 \Omega D\delta w \quad \dots \quad (82)$$

$$[n-\eta(D^2-k^2)]\delta h + h\delta n = H_0 D\delta w \quad \dots \quad (83)$$

$$[n-\eta(D^2-k^2)]\delta\xi + \xi\delta n = H_0 D\zeta. \quad \dots \quad (84)$$

Combining eqs. (71) to (84), we get

$$-\delta n[I_1+(gk^2/n^2)I_2 - I_4 - I_5 - I_7 + I_9] = 2\int_0^d \{n[k^2\rho_0 w - D(\rho_0 Dw)$$
$$+(kH_0/4\pi)(D^2-k^2)Dh_z - \frac{gk^2}{n}(D\rho_0)w - D(2\rho_0\Omega\zeta) + \mu_0(D^2-k^2)^2 w$$
$$+2D\mu_0(D^2-k^2)Dw + D^2\mu_0(D^2+k^2)w\}\delta w dz. \quad (85)$$

We observe that the quantity which appears as a factor of δw under the integral sign on the right-hand side of eq. (85) vanishes if the eq. (31) governing w, h etc. is satisfied. Hence, necessary and sufficient conditions for δn to be zero to the first order for all small arbitrary variations in w, h etc. be solutions of the characteristic value problem. A variational procedure of solving for the characteristic values is therefore possible.

Further properties of n

We interchange i and j in eq. (59), first add and then subtract, the resulting equations become.

$$(n_i+n_j)(I_1+I_4+I_5+I_7+I_9-gk^2/n_in_j)+2\eta k^2(I_4+2I_5+I_6+I_9+I_{10})+2(I_3+I_8) = 0 \quad \dots \quad (86)$$

and

$$(n_i-n_j)(I_1-I_4-I_5-I_7+I_9+gk^2/n_in_j)I_2 = 0. \quad \dots \quad (87)$$

Consider two solutions characterized by n and its conjugate n, we expect, corresponding solutions to be complex conjugates of one another, that is to say if $n_i = n$, $n_j = \bar{n}$, $w_i = w$, $w_j = \bar{w}$, $h_i = h$, $h_j = \bar{h}$, $\zeta_i = \zeta$, $\zeta_j = \bar{\zeta}$, $\xi_i = \xi$ and $\xi_j = \bar{\xi}$.

Substituting in eqs. (86) and (87) we get the following equations,

$$-\mathrm{Re}\,(n) = \frac{(I_3+I_8)+\eta k^2(I_4+2I_5+I_6+I_9+I_{10})}{I_1+I_4+I_5+I_7+I_9-(gk^2/|n|^2)I_2} \quad \dots \quad (88)$$

and

$$\mathrm{Im}(n)[I_1-I_4-I_5-I_7+I_9+(gk^2/|n|^2)I_2] = 0. \quad \dots \quad (89)$$

In the case n is complex $\mathrm{Im}(n) \neq 0$, therefore, its coefficient must vanish, i.e.,

$$[I_1-I_4-I_5-I_7+I_9+(gk^2/|n|^2)I_2] = 0. \quad \dots \quad (90)$$

Combining eqs. (88) and (90), we get

$$-2\,\mathrm{Re}(n) = \frac{(I_3+I_8)+\eta k^2(I_4+2I_5+I_6+I_9+I_{10})}{I_1+I_9}. \quad \dots \quad (91)$$

From which it follows that $\mathrm{Re}(n)$ can never be positive, or in other words, overstability cannot occur.

5. The Case of Exponentially Varying Density

A case for which a simple analytical solution can be found is one in which the undisturbed density distribution is given by

$$\begin{aligned} \rho_0(z) &= \rho_1 \exp \beta z \qquad , 0 < z < d \quad \dots \quad (92) \\ &= 0 \text{ elsewhere} \end{aligned}$$

where ρ_1 and β are constants.

Assuming ν, the coefficient of kinematic viscosity to be constant, we shall take

$$\mu_0(z) = \gamma\rho_1 \exp \beta z. \quad \dots \quad (93)$$

A further assumption, namely that $|\beta d| << 1$ is made, implying that the density variation within fluid is a good deal less than the average density. We shall now consider the case of two free surfaces, so that according to boundary conditions listed in section 3, the following conditions must be satisfied.

$$\begin{aligned} w(0) &= D^2w(0) = Dh(0) = D\zeta(0) = \xi(0) = 0 \\ w(d) &= D^2w(d) = Dh(d) = D\zeta(d) = \xi(d) = 0 \end{aligned} \qquad \ldots \quad (94)$$

Let us assume the following trial functions for $w(z)$, $h(z)$, $\zeta(z)$ and $\xi(z)$ respectively.

$$w(z) = W \sin lz \qquad \ldots \quad (95)$$

$$h(z) = K \cos lz \qquad \ldots \quad (96)$$

$$\zeta(z) = Z \cos lz \qquad \ldots \quad (97)$$

$$\xi(z) = X \sin lz \qquad \ldots \quad (98)$$

where, W, K, Z and X are constant quantities and

$$l = \frac{\pi s}{d}, \quad s \text{ being any integer.} \qquad \ldots \quad (99)$$

Substituting the values of $w(z)$, $h(z)$, $\zeta(z)$ and $\xi(z)$ in eqs. (32), (34) and (35) respectively, we obtain the following equations.

$$n_1 Z - \frac{\kappa H_0 l}{4\pi\rho_1} X = 2\Omega l W \qquad \ldots \quad (100)$$

$$n_2 K = H_0 l W \qquad \ldots \quad (101)$$

$$n_2 X = H_0 l Z \qquad \ldots \quad (102)$$

where

$$n_1 = n + \nu(l^2 + k^2), \quad n_2 = n + \eta(l^2 + k^2) \qquad \ldots \quad (103)$$

Solving the eqs. (100), (101) and (102), we get

$$K = \frac{H_0 l}{n_2} W, \quad X = -\frac{2\Omega H_0 l^2}{n_1 n_2 + V^2 l^2} W, \quad Z = \frac{2\Omega l n_2}{n_1 n_2 + V^2 l^2} W, \qquad \ldots \quad (104)$$

where V denotes the so-called Alfven velocity given by

$$V = (H_0^2/4\pi\rho_1)^{\frac{1}{2}}. \qquad \ldots \quad (105)$$

Evaluating the integrals defined by eqs. (60) to (69) by substituting the values of the trial functions (95) to (98) and inserting them in eq. (70), we obtain after eliminating K^2/W^2, X^2/W^2, Z^2/W^2, on making use of eq. (104),

$$n n_1 (n_1 n_2 + V^2 l^2) - \frac{g\beta k^2}{l^2 + k^2}(n_1 n_2 + V^2 l^2) + n n_1 V^2 l^2 + \frac{n n_2 + 4\Omega l^2}{l^2 + k^2} + \frac{n V^4 l^4}{n_2} = 0 \ldots \quad (106)$$

It reduces to an equation first obtained by Rayleigh (1883) when $\nu = H_0 = \eta = \Omega = 0$. Hide in two separate papers dealt exclusively with the cases (i) $H_0 = 0$, $\eta = 0$ (Hide 1956), (ii) $\Omega = 0$ (Hide 1955).

Ariel (1971) obtained the corresponding equation when $\eta = 0$. The equation (106) can be rewritten as :

$$n^5+2n^4(\eta+\nu)(l^2+k^2)+n^3\left[(l^2+k^2)^2(\eta^2+\nu^2+4\eta\nu)+2l^2V^2+\frac{4\Omega^2l^2-g\beta k^2}{l^2+k^2}\right]$$
$$+n^2[2(\eta+\nu)(l^2+k^2)(l^2V^2+\eta\nu(l^2+k^2)^2)+8\Omega^2l^2\eta-g\beta k^2(2\eta+\nu)]$$
$$+n[(l^2V^2+\eta\nu(l^2+k^2)^2)^2+4\Omega^2l^2\eta^2(l^2+k^2)-\frac{g\beta k^2}{l^2+k^2}l^2V^2-\frac{g\beta k^2}{l^2+k^2}$$
$$\times(2\nu+\eta)(l^2+k^2)-\frac{g\beta k^2\eta}{l^2+k^2}(l^2V^2+\eta\nu(l^2+k^2)^2)=0. \quad \ldots \quad (107)$$

It is convenient to discuss eq. (107) in non-dimensional form, so that the important physical parameters of the problem may be brought out clearly. Let us choose a dimensionless growth rate y and a dimensionless wave number x by measuring n and k in suitable units.

We define :

$$x = \frac{kd}{\pi s}, \qquad y = \frac{nd}{\pi s V}, \quad \ldots \quad (108)$$

so that in dimensionless form eq. (107) becomes

$$y^5+4y^4(R+S)(1+x^2)+y^3\left[4(1+x^2)^2(S^2+R^2+4RS)+2+\frac{A-Bx^2}{1+x^2}\right]$$
$$+2y^2[2(R+S)(1+x^2)(1+4SR(1+x^2)^2)+2AR-Bx^2(2R+S)]$$
$$+y[(1+4RS(1+x^2)^2)^2+4AR^2(1+x^2)-\frac{Bx^2}{1+x}$$
$$-4BR(R+2S)x^2(1+x^2)-2BRx^2(1+4RS(1+x^2)^2)=0, \quad \ldots \quad (109)$$

where

$$A = 4\Omega^2d^2/\pi^2S^2V^2 \quad \ldots \quad (110)$$

$$B = g\beta d^2/\pi^2S^2V^2 \quad \ldots \quad (111)$$

$$S = \pi\nu s/2Vd \quad \ldots \quad (112)$$

$$R = \pi\eta s/2V\,d. \quad \ldots \quad (113)$$

From the above equation, we see that, there are four parameters required to specify y for any given x. These numbers A, B, S and R respectively represent measures of coriolis forces, buoyancy forces, viscous forces and finite conductivity in terms of magnetic field.

Eq. (109) is a quantic in y, hence, it will have five roots. It is too difficult to solve it explicity for arbitrary values of A, B, S and R. However, it can be noted that, if the absolute term in eq. (109) is positive that is $B < 0$, all the terms of eq. (109) are positive. Consequently, either y is real and negative or complex with negative real part. Thus stability occurs for $B < 0$.

Every equation of an odd degree must have atleast one real root having the sign opposite to that of its last term. So that this equation, in case $B > 0$ allows, atleast one root whose real part is positive. In fact, it is the only real root. Accordingly, y or x is real and positive and thus the disturbance grows exponentially with time. The arrangement is therefore unstable for disturbances, corresponding to all wave numbers.

We shall now consider the effect of variation of various parameters on the real value of y which represents the growth rate of disturbance.

First, we observe that

$$y \to \frac{2BR(1+4RS)x^2}{4AR^2+(1+4RS)^2} \qquad (x \to 0) \qquad \dots \quad (114)$$

and

$$\left.\begin{aligned} y &\to \frac{B}{2Sx^2} \qquad \text{if } S \neq 0 \\ y &\to (B)^{\frac{1}{2}} \qquad \text{if } S = 0 \end{aligned}\right\} \qquad (x \to \infty). \qquad \dots \quad (115)$$

Hence we must distinguish between the two cases $S = 0$ and $S \neq 0$, whereas in one former case no mode of maximum instability occurs—y monotonically increases from zero and approaches $(B)^{\frac{1}{2}}$ asymptotically (figure 1), in the latter case there is always a mode of maximum instability which will exert itself early in the motion (figures 2 and 3).

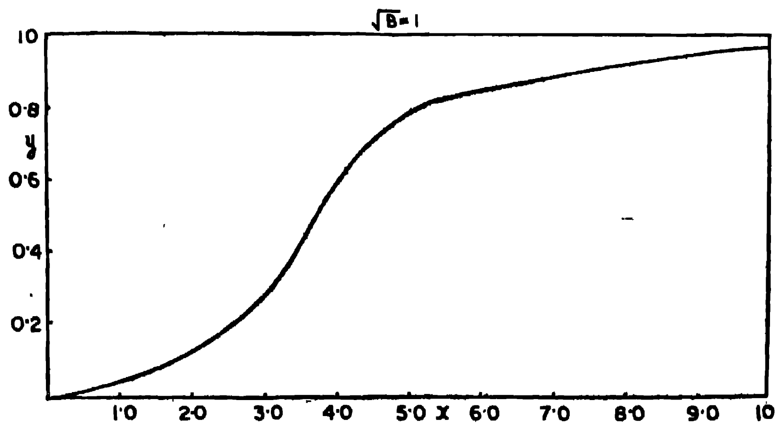

Fig. 1. The growth rate y is plotted as a function of wave number for B = 1, R = 1, S = 0 and for value of A = 10.

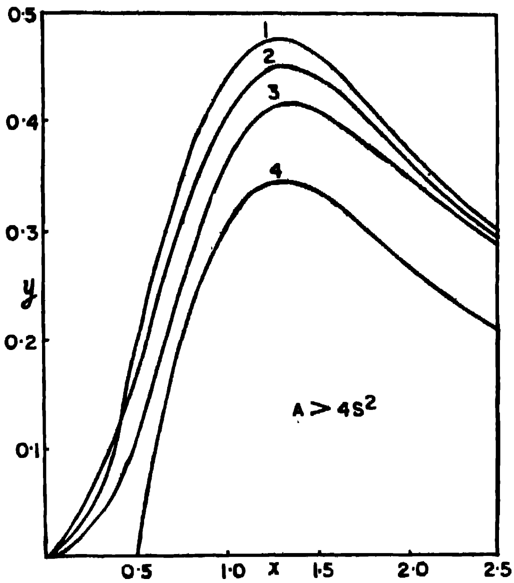

Fig. 2. The growth rate y is plotted as a function of wave number x for $B = 5$, $S = 1$ and $A = 1$. The curve labelled 1, 2, 3 are for values of $R = \infty$, $R = 1$ and $R = 0$.

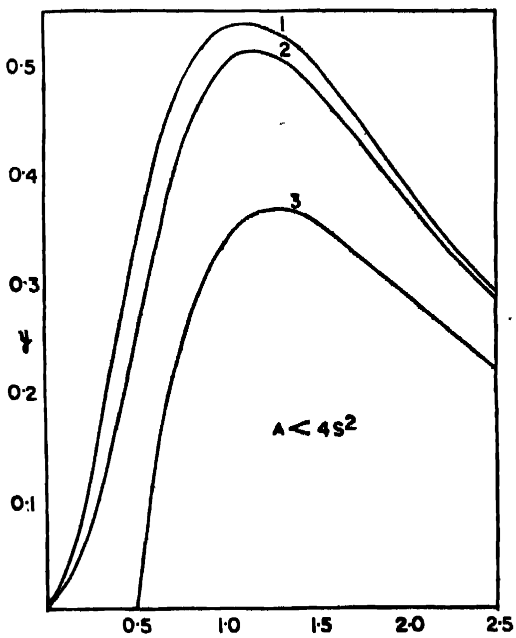

Fig. 3. **Growth rate y is plotted as a function of wave-number x for $B = 5$, $S = 1$ and $A = 10$. The curves labelled 1, 2, 3, 4 are for values of $R = 1$, $R = 0.5$, $R = 0.1$ and $R = 0$.**

However, this distinction between the two cases $S = 0$ and $S \neq 0$ disappears for smaller values of x i.e., for large wave-lengths of disturbance.

We shall first find the behaviour of y on varying the value of R the measure of electrical resistivity. Clearly we must make distinction between the following two cases (i) $A < 4S^2$, (ii) $A > 4S^2$.

(i) $A < 4S^2$. In this case we note that with increasing R, the value of y also increases (figure 2). Thus more the electrical resistivity of the medium, more it tends to destabilize the configuration. It can be further noted that the present case also includes the non-rotating configuration ($A = 0$).

(ii) $A > 4S^2$. In this case a peculiar tendency is exhibited by y as we very the value of R. Two cases arise (a) $R < R^*$ and (b) $R > R^*$ where

$$R^* = \frac{S}{A-4S^2} \qquad \dots \quad (116)$$

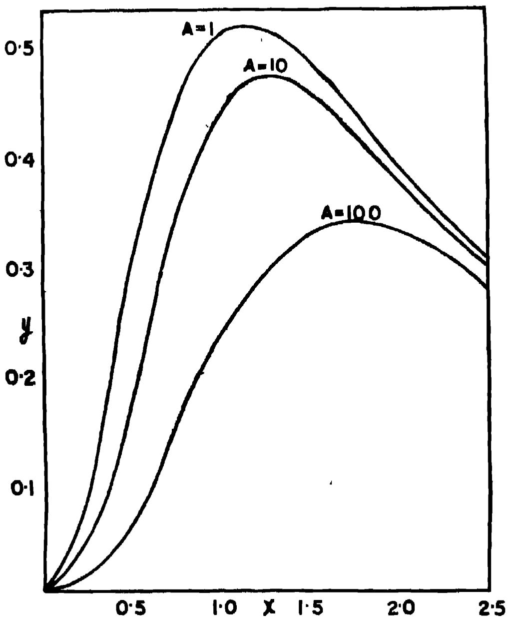

Fig. 4. Illustrates the inhibiting influence of rotation in the unstable case. The growth rate y is plotted as a function of wave number x for B = 5, R = 1, S = 1 and for values of A = 1, A = 10 and A = 100.

So long as $R < R^*$, an increase in the value of R leads to the increase in the value of y and therefore again the system departs faster from one position of equilibrium as we increase the electrical resistivity of the medium. However, when R becomes larger than R^* this behaviour is reversed for small values of x.

Now an increase in the value of R, brings down the value of y for small values of x and thus for larger values of the electrical resistivity the system departs slowly from the position of equilibrium for large wave-lengths of disturbance This peculiar behaviour is however, marked only for small x. After reasonably large values of x, once again the same uniform behaviour can be noted, namely that with increasing electrical conductivity more instability is imparted to the system.

Next we consider the behaviour of variation of the values of A on y. In figure 4 curves of y against x for $R = S = 1$ and $B = 5$ and values $A = 0$, $A = 10$ and $A = 100$ are plotted. It can be seen, (i) that for a given x, y decreases with the increasing A, (ii) that y_m the maximum growth rate, also decreases with increasing A, and (iii) that x_m, the wave number for mode of maximum instability increases with the increasing A. These observations are in agreement with those made by Hide (1955), Talwar (1960) and Ariel (1971).

6. Waves in the Absence of Buoyance Forces

Putting $B = 0$ in eq. (109), we get

$$[y^2+2y(R+S)(1+x^2)+4RS(1+x^2)^2+1]^2 = -\frac{A}{1+x^2}[y+2R(1+x^2)]^2 \quad \ldots \quad (117)$$

If $A = 0$, the eq. (117) becomes

$$y = -(R+S)(1+x^2)\pm[(R-S)^2(1+x_c^2)^2-1]^{\frac{1}{2}}. \quad \ldots \quad (118)$$

From the eq. (118) it can be readily seen that for $0 \leqslant x \leqslant x_c$, where

$$(R-S)^2(1+x_c^2)^2 = 1, \quad \ldots \quad (119)$$

the solution corresponds to damped oscillations. The damping coefficient is given by

$$-R(y) = (R+S)(1+x^2) \quad \ldots \quad (120)$$

and the frequency is

$$I(y) = \pm[1-(R-S)^2(1+x^2)^2]^{\frac{1}{2}}. \quad \ldots \quad (121)$$

Simplifying eq. (117), further, we get

$$y^2+2y\left[(R+S)(1+x^2)+\tfrac{1}{2}i\left(\frac{A}{1+x^2}\right)^{\frac{1}{2}}\right]+4RS(1+x^2)^2+1\pm i2R(1+x^2)^{\frac{1}{2}}A^{\frac{1}{2}} = 0 \quad (122)$$

The solution of eq. (122) can be easily written

$$y = -(R+S)(1+x^2)\pm\tfrac{1}{2}i\left(\frac{A}{1+x^2}\right)^{\frac{1}{2}}\pm\left[(R+S)^2(1+x^2)^2-\frac{A}{4(1+x^2)}\right.$$
$$\left.-4RS(1+x^2)^2-1\pm A^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}(S-R)\right]^{\frac{1}{2}}. \quad \ldots \quad (123)$$

Since in eq. (123) the coefficient of y is complex with non-zero real part, it should have necessarily complex roots. Hence we find that rotation in the absence of buoyancy forces gives rise to damped oscillatory motion throughout the range of wave number x, including the range $x < x_c$ for which the motion is a periodically damped in the absence of coriolis.

$I(y)$, the angular frequency of oscillation, is given by

$$I(y) = \pm\tfrac{1}{2}\left(\frac{A}{1+x^2}\right)^{\frac{1}{2}} \pm \frac{1}{(2)^{\frac{1}{2}}}\left\{\left[\left((R+S)^2(1+x^2)^2 - \frac{A}{4(1+x^2)} - 1\right)^2 + A(1+x^2)(S-R)^2\right]^{\frac{1}{2}} - \left((R+S)^2(1+x^2) - \frac{A}{4(1+x^2)} - 1\right)\right\} \quad \dots \quad (124)$$

The positive and negative signs taken in possible combinations give the angular frequency of four normal modes of oscillations. The expressions for the phase and group velocities can be obtained from eq. (124) with the help of the following relations.

$$U_{p,x} = \pm I(y)/x, \quad U_{q,x} = dI(y)/dx. \quad \dots \quad (125)$$

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